

Classical solutions of two-dimensional Grassmannian models

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Abstract. We present classical solutions of two-dimensional Euclidean Grassmannian σ models (harmonic maps) and discuss some of their properties.

1. INTRODUCTION

It is generally believed that non-Abelian gauge theories are likely to play an important role in any field theoretical description of the theory of elementary particles. In particular, weak and electromagnetic interactions are described by such a theory, and it is generally felt that this is also the case for strong interactions. Gauge theories, in the case of an $SU(2)$ symmetry group, are defined in terms of a Lagrangian density:

$$(1.1) \quad L = \text{tr} F_{\mu\nu} F_{\mu\nu},$$

where

$$(1.2) \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],$$

and where A_μ is an $SU(2)$ valued vector function of a Euclidean four-dimensional space-time.

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One of the main stumbling blocks in making any progress with these theories is our lack of understanding of how to perform functional integrations

$$(1.3) \quad \int D[A_\mu] e^{-\int d^4x \mathcal{L}(A_\mu)} \mathcal{O}(A_\mu)$$

in terms of which most quantities of the theory are given. One line of approach is to attempt to calculate these integrals numerically. As, strictly speaking, functional integrations involve infinite numbers of point integrations, such numerical approaches unavoidably involve various approximations in the form of discretization of the problem, and then further approximations associated with estimating values of the resultant integrals – now finite in number, but still far too many. One resorts to all sorts of sampling techniques, such as Monte Carlo methods, etc., the results of such numerical attempts are encouraging, but so far not conclusive.

When one tries to calculate integrals like (1.3) analytically, one finds that the only viable line of approach available at present is based on the expansion around stationary points of the action and then perturbation theory of the resultant effective theory. To proceed in this way, one has to determine first all stationary points of the action. They are, of course, given by the Euler-Lagrange equations of the theory, which are:

$$(1.4) \quad D_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} - [A_\mu, F_{\mu\nu}] = 0.$$

When written in terms of the gauge potential A_μ , these equations are highly one-linear, second order partial differential equations. As is well known, due to the Bianchi identity

$$(1.5) \quad D_\mu {}^*F_{\mu\nu} = 0$$

where

$$(1.6) \quad {}^*F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}$$

a subclass of solutions of Eqs. (1.4) is provided by the solutions of the first order equations (called «self-duality» equations)

$$(1.7) \quad F_{\mu\nu} = \pm {}^*F_{\mu\nu}.$$

This last equation can be thought of as coming from

$$(1.8) \quad L = \pm q$$

where

$$(1.9) \quad q(x) = \text{tr } F_{\mu\nu} * F_{\mu\nu}$$

is the density of the topological charge.

We are only interested in those solutions of equations of motion for which the action is finite, as it is only for them that we know how to set up the perturbation theory of fluctuations around them. This fact effectively forces A_μ to become pure gauge at infinity and is responsible, in this case, for the existence of the topologically conserved quantity

$$(1.10) \quad Q = \int d^4x q(x).$$

All finite action solutions of Eq. (1.7) have been implicitly, though unfortunately not explicitly, determined by Atiyah, Drinfeld, Hitchin and Manin [1]. In the case of the plus (minus) sign in (1.7), the corresponding finite action solutions are called instantons (anti-instantons). A simple application of a Bogomolnyi bound shows that the instanton and anti-instanton solutions correspond to the local minima of the action. Hence, these solutions are stable under small fluctuations. Even though all finite action solutions of (1.7) were found, it is still not clear whether there are any further solutions of (1.4) which are of finite action, but which are not solutions of (1.7). Some partial negative results exist, but there is no proof, even for the $SU(2)$ theory. Had such non-instanton solutions existed, they presumably would also have had to be included in the stationary point calculation of (1.3). We still do not know.

Moreover, the question of calculating the fluctuations about the instanton solutions has turned out to be a hard mathematical problem. Even in the simplest case of the functional integral (1.3) [$0(A) = 1$], only a partial answer is known. Clearly, not many terms will be calculated in the perturbative scheme mentioned above. Thus it is interesting to know that there exist simple two-dimensional models which bear some resemblance to four-dimensional non-Abelian gauge theories, and in which some of these questions can be partially answered. These models include the CP^{N-1} model originally found by Golo and Perelomov and Eichenherr [2], and also its non-Abelian generalizations [3] (so-called complex Grassmann models). They are essentially free field theories with non-linear geometrical constraints. Many of their properties stem from their geometrical structure. They are known to have an associated linear scattering problem, namely, their equation of motion can be written as an integrability condition of a set of linear equations with a free parameter. They have an infinite set of conservation laws, and these conserved quantities generate an infinite dimensional algebra of dynamical symmetries. Thus, some of their properties may be related

to these additional symmetries of the models and hence not be necessarily a guide to what happens in non-Abelian gauge theories. On the other hand, these models are also interesting per se; in addition, recent work on supergravity has shown that some sectors of this theory are effectively described by σ models very similar to the models we will discuss in the next sections [4]. Moreover, it appears that these models may even be directly relevant in some specific physical cases.

2. $\mathbb{C}P^{N-1}$ MODEL, COMPLEX GRASSMANNIAN σ MODELS AND THEIR INSTANTON SOLUTIONS

Let us denote by $G(M, N)$ the complex Grassmann manifold, which can be written as a quotient space

$$(2.1) \quad G(M, N) = \frac{U(N)}{U(M) \times U(N-M)},$$

where $U(K)$ denotes the group of $K \times K$ unitary matrices. Let $g = g(x)$, where $x = (x_1, x_2) \in R^2$, be an element of $U(N)$ which we decompose as

$$g = (Z, Y)$$

where

$$(2.2) \quad Z = (Z_1, \dots, Z_M), \quad Y = (Z_{M+1}, \dots, Z_N)$$

in which each Z_i ($i = 1, \dots, N$) is an N -component column vector. Then the unitarity of g equivalent to the fact that these vectors are orthonormal to each other

$$(2.3) \quad Z_i \cdot Z_k = \delta_{ik}$$

The Grassmann models are defined by considering the $N \times M$ matrix Z as a dynamical variable, together with the constraint [which comes from (2.3)]

$$(2.4) \quad Z^+ Z = 1_M$$

where 1_M denotes the $M \times M$ unit matrix and $^+$ denotes Hermitian conjugation.

The Lagrangian density and the action of the model are defined as

$$(2.5) \quad L = \text{tr}\{(D_\mu Z)^+ D_\mu Z\}, \quad S = \int d^2x L,$$

where the covariant derivative D_μ is defined by

$$(2.6) \quad D_\mu \Psi = \partial_\mu \Psi - \Psi A_\mu$$

in which A_μ is a composite gauge potential defined by

$$(2.7) \quad A_\mu = Z^+ \partial_\mu Z, \quad A_\mu^+ = -A_\mu.$$

The above-given Lagrangian is invariant under local $U(M)$ transformations

$$Z \rightarrow Z' = Zh \quad \text{where} \quad h = h(x) \in U(M),$$

as then

$$(2.8) \quad D_\mu Z' = (D_\mu Z)h, \quad (D_\mu Z')^+ = h^+(D_\mu Z)^+$$

and also under global $U(N)$ transformations

$$(2.9) \quad Z \rightarrow Z' = gZ, \quad g \in U(N), \quad g = \text{const.}$$

In the special case when $M = 1$, the above model is called the CP^{N-1} model, as the complex Grassmann manifold in this case is equivalent to the complex projective space. The CP^{N-1} model is therefore described by a complex N -component column vector, together with the normalization condition $|Z^2| = Z^+ \cdot Z = 1$. It possesses the Abelian $U(1)$ symmetry and its composite gauge potential A_μ is a pure imaginary function.

The Euler-Lagrange equations of the Grassmannian models, called – in what follows – the equations of motion, are given by

$$(2.10) \quad D_\mu D_\mu Z + Z(D_\mu Z)^+ D_\mu Z = 0$$

together with the constraint (2.4).

As in the gauge theory case, one can introduce a topological charge density

$$(2.11) \quad q(x) = i \epsilon_{\mu\nu} \partial_\mu \text{tr}(Z^+ \partial_\nu Z)$$

and then consider equations coming from the relation (1.8) ($L = \pm q$, selfduality equations). They are given by

$$(2.12) \quad D_\mu Z = \pm i \epsilon_{\mu\nu} D_\nu Z$$

and correspond to Eq. (1.7). Their finite action solutions are again called instantons and anti-instantons respectively. To obtain a better insight into the properties of these models, it is convenient to change the Euclidean variables (x, y) to holomorphic and antiholomorphic variables

$$(2.13) \quad x_\pm = x \pm iy$$

and then rewrite all expressions in terms of these variables. We find

$$(2.14) \quad \begin{aligned} \mathcal{L} &= 2 \text{tr} [(D_+ Z)^+ D_+ Z + (D_- Z)^+ D_- Z] \\ q &= 2 \text{tr} [(D_+ Z)^+ D_+ Z - (D_- Z)^+ D_- Z] \end{aligned}$$

where the derivatives denote the covariant derivative (2.6), in which the differen-

tiation is performed with respect to the x_+ variables. As

$$(2.15) \quad [D_+, D_-] \Psi = \frac{1}{2} \Psi \varrho$$

the equations of motion (2.10) can be rewritten as

$$(2.16a) \quad D_- D_+ Z + Z(D_+ Z)^+ D_+ Z = 0$$

or equivalently

$$(2.16b) \quad D_+ D_- Z + Z(D_- Z)^+ D_- Z = 0$$

and the self-duality equations (2.12) are given by

$$(2.17) \quad D_{\pm} Z = 0.$$

The solutions of these equations in the CP^{N-1} case were already given in the original paper of D'Adda et al. [5]; for the general Grassmannian models they were given by A. Macfarlane [6]. In the CP^{N-1} case they are given by

$$(2.18) \quad Z = f/|f|.$$

where, in the instanton (anti-instanton) case, the f vector is a function of only x_+ (x_-). The finiteness of the action imposes conditions on the components of f – they have to be rational functions of their argument. However, due to the invariance under an overall factor multiplication [due to (2.8)], it is sufficient to consider only polynomial components of f (with no over-all factors). This was found already in the original paper of D'Adda et al. [5], in which a detailed discussion of the instanton (and anti-instanton) solutions was given.

In analogy with (2.18), we find that an instanton solution of the general $G(M, N)$ model is obtained from a set of M linearly independent holomorphic vectors $f_1 \dots f_M$, properly orthonormalized in order to satisfy the constraint (2.4). Let us denote by \hat{Z} the $N \times M$ matrix consisting of the holomorphic vectors $\hat{Z} = (f_1 \dots f_M)$, and define an $M \times N$ Hermitian matrix M by

$$(2.19) \quad M = \hat{Z}^+ \hat{Z}.$$

Because of the linear independence of the vectors M is positive definite and invertible. The Hermitian square root matrices $M^{1/2}$ and $M^{-1/2}$ exist and are unique. Now it is easy to see that

$$(2.20) \quad Z = \hat{Z}(M)^{-1/2}$$

which is a simple generalization of (2.18), satisfies the instanton equations as well as the constraint (2.4).

We see that, in contradistinction to the gauge theory case, the form of all solutions to the first order differential equations (2.17) is very simple and explicit, and this suggests that it may not be too difficult to find finite action solutions of Eq. (2.16) which are not solutions of (2.17).

Such a construction of all finite action solutions of Eq. (2.16) for the CP^{N-1} models and for a large class of solutions of general Grassmannian models will be given in the next sections, where we will also discuss their properties. This construction arose out of a work by Borchers and Garber, who have considered a similar problem in the case of the $O(N)\sigma$ models. This work, with several modifications, could be adapted to the Grassmannian case, where it allows for an elegant mathematical pattern and brings out the geometry of the problem.

Let us finish this section by reformulating the model in a gauge invariant way, the formulation which is often more convenient than the more conventional one discussed so far. To do this, we introduce an $N \times N$ projection matrix P , defined by

$$(2.21) \quad P = Z Z^+ = \sum_{i=1}^M Z_i Z_i^+, \quad P = P^+ = P^2.$$

The Lagrangian takes the form

$$(2.22) \quad L = \text{tr}(\partial_\mu P \partial_\mu P)$$

together with the constraint $P^2 = P$, and the equation of motion is given by

$$(2.23) \quad [\partial_\mu \partial_\mu P, P] = 0.$$

In this formulation, the only difference between different Grassmannian models resides in the rank of P .

Finally, the first order (self-dual) equations (2.17), when written in terms of the projection matrix P , are given by

$$(2.24) \quad \partial_- P \cdot P = 0 \quad \text{and} \quad P \cdot \partial_- P = 0$$

or equivalently

$$(2.25) \quad P \cdot \partial_+ P = 0 \quad \text{and} \quad \partial_+ P \cdot P = 0.$$

3. CP^{N-1} CASE: ISOTROPY AND GENERAL SOLUTIONS

First of all, we consider the CP^{N-1} models, for which we are able to prove some identities (called isotropy conditions by pure mathematicians) which allow us to determine all finite action solutions of the equations of motion. We will then show that similar identities do not necessarily hold in more general

Grassmannian models, thus leaving open the question of the completeness of the derived solutions.

To construct general solutions of the equations of motion, we first study consequences of their existence. If $Z_\alpha(x_+, x_-)$ provides such a finite action solution, then we can show (isotropy condition)

$$(3.1) \quad A_{ij}^m = \overline{D_-^i Z_\alpha} \cdot D_+^j Z_\alpha = 0 \quad \text{for} \quad m = i + j \geq 1.$$

It is clear that $A_{01}^1 = A_{10}^1 = 0$. Moreover, the finiteness of the action and the conservation of energy momentum give $A_{1,1}^2 = 0$. The general case can be proved by induction. We refer the reader, for details of this proof, to Ref. [7]. Here we will state only that one shows first that $\partial_- A_{ij}^m = 0$, and then uses a variant of Liouville's theorem to prove the vanishing A_{ij}^m .

Equation (3.1) shows us that, for a given Z_α , we can construct two subspaces of C^N :

$$(3.2) \quad \begin{aligned} H_k &= \{D_-^i Z_\alpha, \quad i = 1, 2, \dots\} \\ H_m &= \{D_+^j Z_\alpha, \quad j = 1, 2, \dots\} \end{aligned}$$

mutually orthogonal and orthogonal to Z_α . Then one considers the vector space $\hat{H}_K = \{Z_\alpha, H_K\}$ and shows that one can find in it a holomorphic N component vector f_α (i.e., $\partial_- f_\alpha = 0$), expressed as a rational function of Z_α and its derivatives [again, for details, see Ref. [7]]. It is not difficult to show that f_α must be meromorphic; moreover, as solutions to the equations of motion are invariant under the conformal transformation $x_+ \rightarrow 1/x_+$, we see that f_α must be rational.

Now it is possible to show that \hat{H}_K is spanned by $f_\alpha, \partial_+ f_\alpha, \dots, \partial_+^K f_\alpha, \dots$, and that it is possible to express Z_α in terms of this basis. This way we can show that if we write

$$(3.3) \quad Z_\alpha = \frac{\hat{Z}_\alpha}{|\hat{Z}|}$$

$$(3.4) \quad \hat{Z}_\alpha = \partial_+^K f_\alpha - \sum_{i,l=0}^{K-1} M_{il}^{-1} \partial_+ M_{l,K-1} \partial_+^i f_\alpha$$

where

$$(3.5) \quad M_{li} = \overline{\partial_+^l f_\alpha} \cdot \partial_+^i f_\alpha, \quad i, l = 0, 1, \dots, K-1.$$

It now remains to check that indeed Z_α of (3.3) solves the equations of motion. This we can do in several ways, we can show [7] that both Z_α and $D_+ D_- Z_\alpha$ are in H_K and are orthogonal to H_{K-1} – thus they must be proportional:

$$(3.6) \quad D_+ D_- Z_\alpha = \lambda Z_\alpha.$$

Then, a few further lines of algebra show that the z_α solves the equations of motion. Alternatively, we can reformulate our construction, then the proof is even simpler. This we will do in the next sections.

4. ALTERNATIVE FORMULATION. GRAMM-SCHMIDT ORTHONORMALIZATION

Consider a vector $0 \neq g \in C^N$, and define an operator P_+ by

$$(4.1) \quad P_+ g = \partial_+ g - \frac{g(\bar{g}\partial_+ g)}{|g|^2}$$

and define its repeated action as

$$(4.2) \quad P_+^K g = P_+(P_+^{K-1}g).$$

Then it is a matter of algebra to show that \hat{Z}_α of (3.4) is in fact given by

$$(4.3) \quad \hat{Z}_\alpha = (P_+^K f)_\alpha.$$

To proceed further, we note the following useful properties of $P_+^K f$ (when f is an analytic vector):

$$(4.4) \quad \begin{aligned} 1. \quad & \overline{P_+^K f} \cdot P_+^l f = 0 \quad \text{if} \quad l \neq K \\ 2. \quad & \partial_-(P_+^K f) = -P_+^{K-1} f \frac{|P_+^K f|^2}{|P_+^{K-1} f|^2} \\ 3. \quad & \partial_+ \left(\frac{P_+^{K-1} f}{|P_+^{K-1} f|^2} \right) = \frac{P_+^K f}{|P_+^{K-1} f|^2} \\ 4. \quad & P_+^N f = 0. \end{aligned}$$

These properties either follow directly from their definition or are very easy to prove. In proving them, it is useful to introduce wedge products of f and its derivatives. In this way, we define

$$(4.5) \quad h^{(i)} = f \wedge \partial_+ f \wedge \dots \wedge \partial_+^i f \quad i = 0, \dots, N-1$$

and then it is easy to check that

$$(4.6) \quad P_+^K f \sim (h^{K-1})^+ \cdot h^K$$

where the dot product in (4.6) denotes the summation over all indices of h^{K-1}

and all but one of h^K . If f is analytic, so is $h^{(i)}$, although it is an element of a larger-dimensional space. Thus, (4.6) suggests a possible interpretation for the non-instanton ($K \neq 0$) solutions – they correspond to mixtures of instantons and anti-instantons. Both instantons and anti-instantons are elements of larger-dimensional space and are of special form [Eq. (4.5)]. This special form allows their mixtures (4.6) to be elements of C^N , corresponding to the solutions of the equations of motion of the CP^{N-1} model. Thus, in a way, (4.6) shows that all non-linearities of the CP^{N-1} model are associated with the dimension of its manifold and, when properly reinterpreted, the equations of motion are just the Cauchy-Riemann relation for vectors $h^{(i)}$, just as in the instanton case. However, the vectors $h^{(i)}$ have to have their rather specific form (4.5).

The orthogonality properties (4.4) of the $P_+^K f$ vectors show that one can think of them as being obtained from the sequence of holomorphic vectors

$$(4.7) \quad f, \partial_+ f, \partial_+^2 f, \dots, \partial_+^l f, \dots$$

through the Gramm-Schmidt orthogonalization procedure. Moreover, we see that when these vectors are normalized, they provide solutions of the CP^{N-1} model equations of motion. Let us denote by

$$(4.8) \quad e_1, e_2, \dots, e_N$$

the vectors obtained from the sequence (4.7) by the process of Gramm-Schmidt orthonormalization. Then, as Sasaki [8] has shown, one can give a simple proof that any element of the sequence (4.8) solves the CP^{N-1} model equations of motion. To do this, we go to the projector operator formulation of the model. We take, say, the j^{th} element of the sequence and consider the corresponding projector

$$(4.9) \quad P = e_j e_j^+.$$

We also consider another projector

$$(4.10) \quad Q = \sum_{K=1}^{j-1} e_K e_K^+.$$

Then, using relations

$$(4.11a) \quad \partial_- e_l = e_{l-1} (e_{l-1}^+ \partial_- e_l) + e_l (e_l^+ \partial_- e_l)$$

and

$$(4.11b) \quad \partial_+ e_l = e_{l+1} (e_{l+1}^+ \partial_+ e_l) + e_l (e_l^+ \partial_+ e_l)$$

which follow from (4.4), we can establish several identities satisfied by P and Q .

Considering Q as a projector describing an instanton solution of $G(j-1, N)$, we have

$$(4.12) \quad \partial_- Q \cdot Q = 0.$$

In the same way, $P + Q$ describes an instanton solution of $G(j, N)$ and so satisfies

$$(4.13) \quad \partial_- (P + Q) \cdot (P + Q) = 0.$$

However, due to (4.11), we have

$$(4.14) \quad \partial_- P \cdot Q = 0$$

which allows us to derive, from (4.13) and (4.12),

$$(4.15) \quad \partial_- P \cdot P + \partial_- Q \cdot P = 0.$$

However, from (4.11), it also follows that

$$(4.16) \quad P \cdot \partial_+ Q = \partial_+ Q$$

and, by Hermitian conjugation

$$(4.17) \quad \partial_- Q \cdot P = \partial_- Q$$

which permits us to rewrite (4.15) as

$$(4.18) \quad \partial_- P \cdot P + \partial_- Q = 0.$$

If we now take the Hermitian conjugate of (4.18),

$$(4.19) \quad P \cdot \partial_+ P + \partial_+ Q = 0$$

and consider the combination of ∂_+ of (4.18) $- \partial_-$ of (4.19), we obtain the desired equation

$$(4.20) \quad [\partial_+ \partial_- P, P] = 0$$

which completes the proof that the equation of motion is satisfied.

This way of checking whether a given vector satisfies the equations of motion may appear not to be the most convenient choice when CP^{N-1} models are studied; it is, however, very useful for the more general Grassmannian models.

5. GENERAL GRASSMANNIAN MODELS

When we consider a more general Grassmannian model, say, $G(M, N)$, we find several classes of solutions [9]. In particular, we expect that the most generic solutions will depend on M arbitrary analytic vectors $f_1 \dots f_M$. From them, by the generalization of the construction of the previous section, we can construct

the required solution. We proceed as follows. We choose integers $K_i, i = 1, \dots, M$ such that $K_1 \geq K_2 \geq \dots \geq K_M \geq 1$, and such that $\sum_{i=1}^M K_i = N$. Then we consider the N vectors

$$(5.1) \quad \partial_+^i f_i, \quad i = 0, \dots, K_i - 1, \quad i = 1, \dots, M$$

which, except in special cases, will span C^N . We denote these vectors by g_β , $\beta = 1 \dots N$ choosing a certain order. Next we define the subspaces $H_\beta = \{g_1 \dots g_\beta\}$ (with $H_0 = \phi$ and $H_\beta = C^N$ for $\beta \geq N$) and perform a Gramm-Schmidt orthonormalization

$$(5.2) \quad \begin{aligned} \hat{e}_\beta &= g_\beta - g_\beta \downarrow H_{\beta-1}, \quad \beta = 1, \dots, N \\ e_\beta &= \hat{e}_\beta / \|\hat{e}_\beta\|. \end{aligned}$$

Then the statement is that the Grassmannians $Z(\beta), \beta = 1 \dots N - M + 1$, defined by the orthonormal vectors $e_\beta, \dots, e_{\beta+M-1}$ are solutions of the equations of motion provided that the vectors $\{e_\beta\}$ and therefore $\{g_\beta\}$, are chosen in such an order that for all β

$$(5.3) \quad \partial_+ H_\beta \subset H_{\beta+M}.$$

This condition is fulfilled, for example, by the choice $\{g_\beta\} = \{f_1, \dots, f_M, \partial_+ f_1, \partial_+ f_2, \dots, \partial_+ f_M, \partial_+^2 f_1, \dots\}$ but, of course, other permutations of this particular order also fulfil (5.3), such as starting the sequence $\{g_\beta\}$ with a certain number of derivative vectors $\partial_+^i f_1$ and then continuing with f_2, f_3, \dots and further on with $\partial_+^j f_1$, etc. To show that the Grassmannian $Z(\beta)$ satisfies the equations of motion, one introduces P , the projector on $H_{\beta+M-1}/H_{\beta-1}$, and Q , the projector on $H_{\beta-1}$. One then repeats all the arguments of the last part of the last section [observing on the way that Q describes an instanton of $G(\beta - 1, N)$ and $P + Q$ an instanton of $G(\beta + M - 1, N)$].

Having found these solutions of the equations of motion based on M arbitrary analytic vector functions, we may ask whether there are further solutions of the $G(M, N)$ models. It turns out that the answer to this question is positive: the general structure of these further solutions is such that they are all determined in terms of a smaller number of arbitrary holomorphic (with polynomial components) $f_1, \dots, f_{M'}$, with $M' < M$. The vectors of basis $\{g_\beta\}$ are so chosen that $\partial_+ H_\beta \subset H_{\beta+M'}$, for any β . Then one can form an ordered set of vectors e_β , constructed in the same way as before and then partition this ordered set into a sequence of subsets, each of length at least M' (with the exception of perhaps the first one). Then, taking M vectors which are given by a collection of such subsets gives a Grassmannian which satisfies the equations of motion of the $G(M, N)$ model. One can easily give examples of such special solutions. For

instance, if we take $M' = 1$, then all vectors of the basis are given in terms of one vector f . Then $\partial_+ H_\beta \subset H_{\beta+1}$ and any M distinct vectors from the sequence $\{e_\beta\}$ provide a solutions. In the case of two holomorphic vectors f_1, f_2 with, say, $g_\beta = \{f_1, f_2, \partial_+ f_1, \partial_+ f_2, \dots\}$, i.e., such that $\partial_+ H_\beta \subset H_{\beta+2}$, the sets $\{e_1, e_2, e_6, e_7\}$ or $\{e_3, e_4, e_7, e_8\}$ are solutions corresponding to $M = 4$.

To prove that the equations of motion are satisfied, we observe that if we denote the corresponding projector by P and the complementary projector by Q (i.e., such that $P + Q$ is a projector on the surface spanned by all the vectors of the basis up to the last vector in the sequence defining the Grassmannian), we have

$$(5.4) \quad \begin{aligned} \partial_-(P + Q) \cdot (P + Q) &= 0 \\ Q \cdot P &= P \cdot Q = 0 \end{aligned}$$

and also

$$(5.5) \quad \begin{aligned} (P + Q) \cdot \partial_+ Q &= \partial_+ Q \\ \partial_-(P + Q) \cdot P &= 0 \end{aligned}$$

from

$$(5.6) \quad \partial_+ H_\beta \subset H_{\beta+M'}$$

Thus

$$(5.7) \quad \partial_- P \cdot P + \partial_- Q - \partial_- Q \cdot Q = 0$$

and we see that

$$(5.8) \quad [\partial_+ \partial_- P, P] = [\partial_+ \partial_- Q, Q]$$

and we find that P is a solution if Q is also one. To prove that Q is a solution, we repeat this prescription, this time treating Q as our previous P and introducing a new complementary Q' . We find

$$(5.9) \quad [\partial_+ \partial_- Q, Q] = [\partial_+ \partial_- Q', Q']$$

thus showing that the problem is reduced to proving that Q' satisfies the equations of motion (of a corresponding Grassmannian). It is easy to convince oneself that the final Q in this chain of steps describes an instanton solution, proving that all intermediate Q 's (and P 's) are also solutions of the corresponding equations of motion (notice that Q 's and P 's in general describe Grassmannians corresponding to different M 's).

Have we then found all solutions of the equations of motion? Can we repeat the proof of completeness (given beforehand for the CP^{N-1} model)? In that

proof, we first of all showed that the spaces spanned by $D_+^i Z$ and $D_-^i Z$ were orthogonal (3.1). A few lines of algebra show that as the covariant derivatives involve non-Abelian gauge fields, the previous proof does not go through. Moreover, a glance at the solutions we have found shows that

$$(5.10) \quad A_{ij}^m = (D_-^i Z)^+ \cdot D_+^j Z \neq 0 \quad \text{for} \quad m = i + j \geq 1$$

for some of them. In fact, A_{ij}^m vanishes for all generic solutions, but does not do so for some special solutions. This is true, for example, for solutions based on one vector f [where the size of the «gap» between subsets is related to m in (5.10), for which $A_{ij}^m \neq 0$]. However, it is easy to check that although $A_{ij}^m \neq 0$, their trace vanishes:

$$(5.11) \quad \text{tr} A_{ij}^m = 0 \quad \text{for} \quad m = i + j \geq 1.$$

Can we show that this is a consequence of the finiteness of the action? It is easy to show that

$$(5.12) \quad \text{tr} A_{i+1j}^{m+1} = -\text{tr} A_{ij+1}^{m+1}$$

but so far we have not succeeded in showing that $\text{Tr} A_{ij}^m = 0$. Thus it is possible that this condition does not have to be satisfied, in which case there exist further classes of solutions, in addition to the ones discussed above. We hope that this question can be resolved one way or another in the not-too-distant future.

6. PROPERTIES OF THE SOLUTIONS-ACTION AND THE TOPOLOGICAL CHARGE

We start considering properties of the solutions by evaluating the values of the action and of the topological charge. The expressions for their densities are given by

$$(6.1) \quad S = 2 \text{tr} [(D_+ Z)^+ D_+ Z + (D_- Z)^+ D_- Z]$$

and

$$(6.2) \quad q = 2 \text{tr} [(D_+ Z)^+ D_+ Z - (D_- Z)^+ D_- Z].$$

We shall first prove that

$$(6.3) \quad \text{tr} \{ [D_+ Z(\beta)]^+ D_+ Z(\beta) \} = \text{tr} \{ [D_- Z(\beta + M)]^+ D_- Z(\beta + M) \}$$

which is a natural generalization of

$$(6.4) \quad \left| D_- \left(\frac{p_+^{K+1} f}{|p_+^{K+1} f|} \right) \right|^2 = \left| D_+ \left(\frac{p_+^K f}{|p_+^K f|} \right) \right|^2$$

found before for the CP^{N-1} model.

To show (6.3), we consider

$$(6.5) \quad P = Z(\beta)Z(\beta)^+ = \sum_{i=\beta}^{\beta+M-1} e_i e_i^+,$$

$$P' = Z(\beta+M)Z(\beta+M)^+ = \sum_{i=\beta+M}^{\beta+2M-1} e_i e_i^+$$

and

$$Q = \sum_{i=1}^{\beta-1} e_i e_i^+,$$

Then, as

$$(6.6) \quad \begin{aligned} D_+ Z &= \partial_+ P \cdot Z \\ (D_- Z)' &= \partial_- P' \cdot Z' \end{aligned}$$

we find that the left-hand side of (6.3) is given by

$$(6.7) \quad L = \text{tr}[(\partial_+ PZ)^+ \partial_+ PZ] = \text{tr}[P \cdot \partial_- P \cdot \partial_+ P] = \text{tr}[(\partial_+ P \cdot P)^+ \partial_+ PP]$$

while the right-hand side is given by

$$(6.8) \quad R = \text{tr}[(\partial_- P'Z')^+ \partial_- P'Z'] = \text{tr}[P' \cdot \partial_+ P' \cdot \partial_- P'] = \text{tr}[(\partial_- P' \cdot P')^+ \partial_- P' \cdot P'].$$

However, using the expression (4.18) applied to $\partial_- P'P'$ and then to $\partial_- PP$,

$$(6.9) \quad \partial_- P' \cdot P' = -\partial_- (P + Q) = -\partial_- P - \partial_- Q = -\partial_- P + \partial_- PP = -P \partial_- P$$

we find that

$$(6.10) \quad R = \text{tr}[(P \cdot \partial_- P)^+ P \cdot \partial_- P] = \text{tr}[\partial_+ P \cdot P \cdot \partial_- P] = \text{tr}(P \cdot \partial_- P \cdot \partial_+ P) = L$$

thus proving (6.3).

Having proved (6.3), we show that the problem of calculating the action has been reduced to that of determining the topological charge. To see this, we look first at the case $M = 2$. There, introducing the notation

$$(6.11) \quad |D_{\pm} Z(\beta)|^2 = \text{tr} \{ [D_{\pm} Z(\beta)]^+ [D_{\pm} Z(\beta)] \}$$

we find

$$(6.12) \quad \begin{aligned} S &= |D_+ Z(\beta)|^2 + |D_- Z(\beta)|^2 = |D_+ Z(\beta)|^2 - |D_- Z(\beta)|^2 + \\ &+ 2|D_+ Z(\beta-2)|^2 - 2|D_- Z(\beta-2)|^2 + 2|D_+ Z(\beta-4)|^2 - \\ &- 2|D_- Z(\beta-4)|^2 + \dots + 2|D_+ Z(p)|^2, \end{aligned}$$

where $p = 2$ if β is even or $p = 1$ if β is odd. However, as

$$(6.13) \quad |D_- Z(p)|^2 = 0$$

for $p = 1, 2$, we see that we can rewrite (6.12) as

$$(6.14) \quad S(\beta) = q(\beta) + 2q(\beta - 2) + 2q(\beta - 4) + \dots + 2q(p)$$

and in the general case (for arbitrary M)

$$(6.15) \quad S(\beta) = q(\beta) + 2q(\beta - M) + 2q(\beta - 2M) + \dots + 2q(p),$$

where $p = \beta \pmod{M}$, or M if $\beta \pmod{M} = 1$.

To calculate the topological charge of the configuration

$$(6.16) \quad Z = (e_{\beta}, e_{\beta+1}, e_{\beta+2}, \dots, e_{\beta+M-1})$$

we observe that, using the cyclic permutation property of the trace, we can write:

$$(6.17) \quad q_{\pm} = 2[\text{tr}\{(D_{\pm} Z)^+ D_{\pm} Z - (D_{\pm} Z)^+ D_{\pm} Z\}] = 2\text{tr}((\partial_{+} Z)^+ \partial_{+} Z - \partial_{+} Z)^+ \partial_{+} Z).$$

Then it is a matter of algebra to show that

$$(6.18) \quad q_{\pm} = \sum_{i=\beta}^{\beta+M-1} q_i, \quad q_i = 2 \partial_{+} \partial_{-} \ln |\hat{e}_i|^2.$$

As expected, the topological charge density is completely additive. This property allows us to rewrite the action density corresponding to the general case as

$$(6.19) \quad S(\beta) = 2q_1 + 2q_2 + \dots + 2q_{\beta-1} + q_{\beta} + q_{\beta+1} + \dots + q_{\beta+M-1}.$$

Next we calculate the integrated values of the topological charge and of the action. As

$$(6.20) \quad \partial_{+} \partial_{-} \ln |p|^2 = \frac{1}{4} \partial_{\mu} \partial_{\mu} \ln |p|^2$$

the use of the divergence theorem in two dimensions shows that

$$(6.21) \quad \int d^2x \partial_{+} \partial_{-} \ln |p|^2 = \pi \alpha$$

where $|p| \rightarrow |x|^{\alpha}$ as $|x| \rightarrow \infty$, and where we have assumed that $|p|$ has no singularities (and no zeros) except those at ∞ .

To calculate the asymptotic behaviour of $|\hat{e}_i|^2$, it is convenient to use the wedge product formalism [of (4.5) and (4.6)]. We let, using vectors of the basis,

$$(6.22) \quad h^{(i)} = g_1 \wedge g_2 \wedge g_3 \wedge \dots \wedge g_i,$$

and then

$$(6.23) \quad \hat{e}_\beta \sim (h^{\beta-1})^+ \cdot h^\beta.$$

Moreover, it is easy to see that up to an overall constant factor

$$(6.24) \quad |\hat{e}_\beta|^2 \cong |h^\beta|^2 / |h^{\beta-1}|^2.$$

Then, defining

$$(6.25) \quad \begin{aligned} \alpha_0 &= 0 \\ \alpha_\beta &= \deg h^\beta \text{ mod (overall factors)} \end{aligned}$$

we find that [9]

$$(6.26) \quad Q_i = \int d^2x q_i = 2\pi(\alpha_i - \alpha_{i-1})$$

thus showing that the topological charge and the action of the field (6.16) are given by

$$(6.27) \quad \begin{aligned} Q &= 2\pi(\alpha_{\beta+M-1} - \alpha_{\beta-1}) \\ S &= 2\pi(\alpha_{\beta+M-1} + \alpha_{\beta-1}). \end{aligned}$$

These are natural generalizations of the corresponding results, previously obtained in the CP^{N-1} model.

All this discussion concerned the generic solutions. For the special solutions, the discussion is very similar, except that one has to pay special attention as to whether the vectors which form Z are all adjacent or fall into groups of adjacent vectors separated by «gaps». Each group of adjacent vectors (separated by «gaps») is treated as a separate unit. For its vectors, the results of (6.27) (with appropriately modified indices β and M) still hold. Then the total topological charge and the action for the Grassmannian are just sums of such expressions for each group.

The non-instanton (and non-anti-instanton) solutions are characterized by $|Q| \neq S$, and so for them the usual topological arguments guaranteeing their stability do not apply. Thus they may not correspond to local minima of the action, and in fact, this is indeed the case. They are unstable; there exist modes of fluctuations around them for which the action decreases. To see this, we take a general solution Z for which $D_\pm Z \neq 0$, and consider a small fluctuation ϕ around Z . As a result of this fluctuation, the Z field is modified to

$$(6.28) \quad Z' = Z\sqrt{1 - \phi^+ \cdot \phi} + \phi,$$

where $\phi^+ Z = 0$ [one can always describe the general fluctuation in the form, due to the invariance of the theory under $U(M)$ transformations]. The action for the modified field is given by

$$(6.29) \quad S' = 2tr \int d^2x [(D'_+ Z')^+ \cdot D'_+ Z' + (D'_- Z')^+ \cdot D'_- Z'],$$

where D'_i are the usual covariant derivatives written in terms of Z' . Since the topological charge is invariant under small fluctuations, we can rewrite this as

$$(6.30) \quad \begin{aligned} S' &= 2 \int d^2x q(x) + 4 \int d^2x tr[(D'_- Z')^+ D'_- Z'] = \\ &= S + 4 \int d^2x V(\phi), \end{aligned}$$

where $V(\phi)$ calculated to second order in the small fluctuation ϕ is given by.

$$(6.31) \quad \begin{aligned} V(\phi) &= tr[(D_- \phi)^+ D_- \phi] - tr[\phi^+ \cdot \phi (D_- Z)^+ D_- Z] \\ &\quad - tr[(Z^+ \cdot D_- \phi + \phi^+ \cdot D_- Z)^+ \cdot (Z^+ \cdot D_- \phi + \phi^+ \cdot D_- Z)] \end{aligned}$$

[or we can rewrite $V(\phi)$ in terms of expressions involving D_+ only]. Now, if we choose

$$(6.32) \quad \phi = \epsilon D_+ Z$$

where ϵ is a constant complex number, we can show that for our solutions

$$(6.33) \quad \begin{aligned} V &= -|\epsilon|^2 tr[(D_+ Z)^+ \cdot D_+ Z \cdot (D_- Z)^+ \cdot D_- Z] - \\ &\quad - |\epsilon|^2 tr[(D_- Z)^+ \cdot D_+ Z \cdot (D_+ Z)^+ \cdot D_- Z] \end{aligned}$$

and so is negative definite if neither $D_- Z$ nor $D_+ Z$ vanish. In deriving this result, we used the equations of motion and also the property that

$$(3.34) \quad tr[(D_- Z)^+ \cdot D_+ Z \cdot (D_+ Z)^+ \cdot D_+ Z] = 0$$

for our solutions. This result is trivial for the CP^{N-1} case and for the generic solutions (where not only the trace but the whole matrix vanishes). For the special cases, it is true as well, although this time it follows from the fact that for our special solutions the following property holds – if an element of

$D_- Z^+ D_+ Z$ is non-zero, the corresponding element of the Hermitian matrix $D_+ Z^+ D_+ Z$ vanishes. This guarantees that although $D_- Z^+ D_+ Z D_+ Z^+ D_+ Z \neq 0$, its trace vanishes. It is interesting that the proof of the relations like (6.34) is required if one wants to derive a proof of completeness of solutions. We see that as $S' < S$, all non-instanton solutions are unstable, i.e., they correspond to the saddle points of the action.

7. BACKGROUND FERMION PROBLEM

Next, we briefly consider solutions of the Dirac-like equations in the background of the Grassmannian fields

$$(7.1) \quad \not{D}\psi - ZZ^+ \not{D}\psi = 0,$$

where the fermionic field ψ , like Z , is given by an $N \times M$ matrix, and satisfies

$$(7.2) \quad Z^+ \cdot \psi = 0$$

(this form of the equation results from the reduction of the supersymmetric form of the problem).

It is convenient to resolve ψ into eigenstates of γ_5 :

$$(7.3) \quad \psi = \begin{pmatrix} 1 \\ -i \end{pmatrix} \psi^+ + \begin{pmatrix} 1 \\ i \end{pmatrix} \psi^-.$$

Then one obtains the equivalent equations

$$(7.4) \quad D_{\pm} \psi^{\pm} = Z \lambda_{\pm}$$

together with the constraints

$$(7.5) \quad Z^+ \cdot \psi^{\pm} = 0$$

where λ_{\pm} are $M \times M$ matrix-valued functions of x_+ and x_- . To solve these equations, it is convenient to use the expression for the Grassmannian $Z(\beta)$, whose column vectors are not mutually orthogonal. In this case, having chosen β , we consider the M vectors $g_{\beta}, g_{\beta+1}, \dots, g_{\beta+M-1}$, and orthogonalize them with respect to $H_{\beta-1}$ (but not to each other), defining

$$(7.6) \quad \hat{g}_j = g_j - g_j \downarrow H_{\beta-1}; \quad j = \beta, \beta + 1, \dots, j + M - 1.$$

Then, having formed a matrix \hat{Z} , whose columns are given by g_j , we find that the Grassmannian $Z(\beta)$ is given by

$$(7.7) \quad Z = \hat{Z}(M)^{-1/2},$$

where

$$(7.8) \quad M = \hat{Z}^+ \hat{Z}.$$

This expression is, of course, completely equivalent to (5.2), but it happens to be more convenient in the fermion background problem.

To solve (7.4), we observe that

$$(7.9) \quad \begin{aligned} D_+ M^{1/2} &= \partial_+ M^{1/2} - M^{1/2} M^{-1/2} \hat{Z} \partial_+ (\hat{Z} M^{-1/2}) - \\ &- \hat{Z}^+ \partial_+ (\hat{Z} M^{-1}) M^{1/2} = 0 \end{aligned}$$

and similarly

$$(7.10) \quad D_- M^{-1/2} = 0.$$

Thus, setting

$$(7.11) \quad \psi^\pm = \phi_\pm M^{\pm 1/2},$$

we find that as

$$(7.12) \quad \begin{aligned} D_\pm (\phi_\pm M^{\pm 1/2}) &= \partial_\pm (\phi_\pm M^{\pm 1/2}) - \phi_\pm M^{\pm 1/2} M^{-1/2} \hat{Z}^+ \cdot \\ &\cdot \partial_\pm (\hat{Z} M^{-1/2}) = (\partial_\pm \phi_\pm) M^{\pm 1/2} \end{aligned}$$

our equations (7.4) and (7.5) are equivalent to

$$(7.13) \quad \begin{aligned} \partial_\pm \phi_\pm &= Z \mu_\pm \\ Z^+ \cdot \phi_\pm &= 0, \end{aligned}$$

where μ_\pm are some $M \times M$ matrix valued functions of x_+ and x_- .

To solve these equations, we use some basic properties of the projectors \mathbb{P}_β on the subspace $H_\beta = \{g_1, \dots, g_\beta\}$. Let us denote by P the projector on the subspace defined by

$$(7.14) \quad Z = (e_{\beta^+}, e_{\beta^++1}, \dots, e_{\beta^++M-1})$$

(i.e., $P = \mathbb{P}_{\beta^++M-1} - \mathbb{P}_{\beta^+-1}$).

Then

$$(7.15) \quad \partial_+ \mathbb{P}_{\beta^+-1} = -P \cdot \partial_+ P$$

(which follows from discussions of Section 4). If we now take

$$(7.16) \quad \phi_+ = \mathbb{P}_{\beta^+-1} \cdot h^+(x_-)$$

where h^+ is an $N \times M$ matrix dependent on x_- , we see that the constraint is automatically satisfied and that

$$(7.17) \quad \partial_+ \phi_+ = (\partial_+ \mathbb{P}_{\beta^+-1}) h^+ = -P \cdot \partial_+ P \cdot h^+$$

which shows that the equation (7.13) is automatically satisfied with

$$(7.18) \quad \mu_+ = -Z^+ \cdot \partial_+ P \cdot h^+.$$

In a similar way, we find that

$$(7.19) \quad \phi_- = (\mathbb{1} - \mathbf{P}_{\beta+M-1}) \cdot h^-(x_+)$$

solves the other equation.

The derived expressions for ψ^+ and ψ^- can then be used to check the Atiyah-Singer index theorem. This theorem relates the number of independent solutions of the background Dirac equation (normalized on a sphere) to the topological properties of the background field, and is given by

$$(7.20) \quad \xi_+ - \xi_- = NQ$$

(where ξ_{\pm} denotes the number of helicity \pm solutions). In the instanton (anti-instanton) case, the condition of normalizability on the sphere makes ξ_- (ξ_+) vanish. In the general case, neither number vanishes, and so the theorem is satisfied in the non-minimal way.

It is interesting to observe that the solutions of the fermion background problem provide us with examples of negative modes of the fluctuation operator discussed in the previous section, and as such, can be used to derive lower bounds on the number of such modes.

To see this, let us consider Z given by (7.14), and take as a fluctuation ϕ

$$(7.21) \quad \phi = \mathbf{P}_{\beta-1} h^+(x_-) M^{1/2}.$$

Then choosing to write $V(\phi)$ in (6.31) in terms of D_+ derivatives

$$(7.22) \quad \begin{aligned} V(\phi) = & \text{tr}[(D_+ \phi)^+ D_+ \phi] - \text{tr}[\phi^+ \phi (D_+ Z)^+ D_+ Z] - \\ & - \text{tr}[(Z^+ \cdot D_+ \phi + \phi^+ \cdot D_+ Z)^+ \cdot (Z^+ \cdot D_+ \phi + \phi^+ \cdot D_+ Z)] = \text{tr}[(D_+ \phi)^+ \cdot \\ & \cdot D_+ \phi] - \text{tr}[\phi^+ \phi (D_+ Z)^+ D_+ Z] - \text{tr}[(Z^+ \cdot D_+ \phi)^+ Z^+ D_+ \phi], \end{aligned}$$

as

$$\phi^+ \cdot D_+ Z = M^{1/2} (h^+)^+ \mathbf{P}_{\beta-i} \partial_+ P \cdot Z = 0.$$

But

$$(7.23) \quad \begin{aligned} & \text{tr}[(D_+ \phi)^+ D_+ \phi] - \text{tr}[(Z^+ \cdot D_+ \phi)^+ Z^+ \cdot D_+ \phi] = \\ & = \text{tr}[(D_+ \phi)^+ \cdot (1-P) \cdot D_+ \phi] = 0 \end{aligned}$$

as

$$D_+ \phi = -P \cdot \partial_+ P \cdot h^+ \cdot M^{1/2}$$

due to (7.16). Thus we see that

$$(7.24) \quad V(\phi) = -\operatorname{tr}[\phi^+ \phi(D_+ Z)^+ D_+ Z] \leq 0.$$

In a similar way, we can show that also

$$(7.25) \quad V(\phi = (1 - \mathbf{P}_{\beta^+ M-1}) h^-(x_+) M^{-1/2}) \leq 0$$

The conditions on functions h^+ and h^- are weaker than in the background Dirac problem case – now we require that

$$|\phi|^2 < 1.$$

8. RELATION TO THE HILBERT-RIEMANN PROBLEM

As the Grassmann models are completely integrable, one can analyse them from the point of view of the associated Hilbert-Riemann problem. In this approach, one introduces a linear system of equations for a $N \times N$ matrix-valued function $\Psi(x_+, x_-, \lambda)$, where λ is an additional complex parameter, such that the integrability conditions for ψ are equivalent to the equations of motion for the non-linear problem in question. In our case, the linear equations are [10]:

$$(8.1) \quad \begin{aligned} \partial_+ \psi &= \frac{2}{1+\lambda} [\partial_+ P, P] \psi \\ \partial_- \psi &= \frac{2}{1-\lambda} [\partial_- P, P] \psi, \end{aligned}$$

where P denotes the projector describing the Grassmannian ($P = ZZ^+$). The integrability condition of the system (8.1) is then given by

$$(8.2) \quad [\partial_+ \partial_- P, P] = 0$$

i.e., by the Grassmannian equations of motion. Let us consider first the (\mathcal{P}^{N-1}) model case, and let us take the solution $e_\beta = P_+^{\beta-1} f / |P_+^{\beta-1} f|$ and denote by P_β the corresponding projector

$$(8.3) \quad P_\beta = e_\beta e_\beta^+.$$

Then the solution of the system (8.1) is given by

$$(8.4) \quad \psi = \mathbb{1} + \frac{4\lambda}{(\lambda-1)^2} \sum_{\alpha=1}^{\beta-1} P_\alpha + \frac{2}{\lambda-1} P_\beta,$$

which, using $\sum_{\beta=1}^N P_\beta = 1$ can be rewritten in two equivalent forms (up to over-

all factors)

$$\begin{aligned}
 \psi &= \mathbb{1} - \frac{4\lambda}{(\lambda+1)^2} \sum_{\alpha=\beta+1}^N P_\alpha - \frac{2}{\lambda+1} P_\beta = \\
 (8.5) \quad &\cong \mathbb{1} + \frac{2}{\lambda-1} \sum_{\alpha=1}^{\beta-1} P_\alpha - \frac{2}{\lambda+1} \sum_{\alpha=\beta+1}^N P_\alpha.
 \end{aligned}$$

These general ψ solutions thus exhibit either simultaneous first-order poles at $\lambda = 1$ and $\lambda = -1$, or up to second-order poles at $\lambda = 1$ or alternatively at $\lambda = -1$.

For the case of the Grassmannians, the form of the solutions depends on whether the solutions are generic or special. For the generic solution described by $Z(\beta)$, we have $P = \sum_{\alpha=\beta}^{\beta+M-1} P_\alpha$ where $P_\alpha = e_\alpha e_\alpha^+$ and the corresponding solutions for ψ are again given by

$$(8.6) \quad \psi = \mathbb{1} + \frac{4\lambda}{(\lambda-1)^2} \sum_{\alpha=1}^{\beta-1} P_\alpha + \frac{2}{\lambda-1} P$$

in complete analogy to (8.4). This expression can also be rewritten in forms similar to (8.5). For the special solutions, the situation is more complicated. Let us discuss, as an illustration, the case of $M = 2$ and look at solutions based on one vector f (i.e., $g_\beta = \partial_+^{\beta-1} f$). Then for $\beta_2 > \beta_1 + 1$

$$(8.7) \quad P = P_{\beta_1} + P_{\beta_2} \quad \text{where} \quad P_\alpha = e_\alpha e_\alpha^+$$

is a solution of the equations of motion discussed in Section 5. The corresponding solution of the *HR* problem is given by

$$\begin{aligned}
 \psi &= 1 + \frac{8\lambda(\lambda^2+1)}{(\lambda-1)^4} \sum_{\alpha=1}^{\beta_1-1} P_\alpha + \frac{2(3\lambda^2+1)}{(\lambda-1)^3} P_{\beta_1} + \frac{4\lambda}{(\lambda-1)^2} \sum_{\alpha=\beta_1+1}^{\beta_2-1} P_\alpha + \\
 (8.8) \quad &+ \frac{2}{\lambda-1} P_{\beta_2} \cong 1 + \frac{4\lambda}{(\lambda-1)^2} \sum_{\alpha=1}^{\beta-1} P_\alpha + \frac{2}{\lambda-1} P_{\beta_1} - \frac{2}{\lambda+1} P_{\beta_2} - \\
 &- \frac{4\lambda}{(\lambda+1)^2} \sum_{\alpha=\beta_2+1}^N P_\alpha
 \end{aligned}$$

and we note that this solution exhibits poles of either up to second order simultaneously at $\lambda = 1$ and $\lambda = -1$, or up to fourth order at $\lambda = +1$ or alternatively at $\lambda = -1$. For $m \geq 2$, it is easy to display special types of solutions which

lead to still higher-order poles at $\lambda = \pm 1$ in ψ . The order of these poles is related to the number of «gaps» in the group of vectors used in the construction of the solution.

Let us point out at this stage that the technique of the *RH* problem, as usually applied to non-linear equations, involves an ansatz for ψ in terms of only first-order poles. As seen above, the example of $G(M, N)$, where relations are known by other techniques, shows that a more complicated ansatz may lead to different and interesting kinds of solutions.

Let us finish this section by noting that in the traditional approach to the *HR* problem, one tries to determine new solutions from the old ones, and attempts at the same time to determine new field configurations which would automatically solve their non-linear equations of motion. This is done by setting

$$(8.9) \quad \psi^{\text{new}}(\lambda) = \chi(\lambda) \psi^{\text{old}}(\lambda)$$

and then trying to solve the resultant equation for $\chi(\lambda)$. When we specialize this procedure to our case, and take for $\psi^{\text{old}}(\lambda)$ some known solutions discussed above, we do obtain linear equations for $\chi(\lambda)$; however, they seem to be as hard to solve as the original equations for $\psi(\lambda)$. No particular simplification results from following the traditional line of approach based on Eq. (8.9).

9. CONCLUSIONS AND OPEN QUESTIONS

Clearly, although some understanding of the structure of the classical solutions has been found, there are still many unanswered questions. As far as the classical solutions are concerned, the main outstanding question is the completeness of the solutions in more general Grassmannian cases. Then one would like to understand quantum corrections to these classical results. Of course, the instability of the solutions, and hence the existence of negative modes, complicates the discussion, but presumably one should be able to find an appropriate analytical continuation. However, this has turned out to be quite difficult. It is very difficult to determine the spectrum of the fluctuation operator (or even be certain as to the exact number of negative modes), and in one simple case, when the spectrum is known, it seems rather difficult to deal with zero modes.

In the approximation of considering only instanton and anti-instanton solutions, Berg and Lüscher [11] and Fateev, Frolov and Schwarz [12] showed that the quantum corrections to the classical solutions can be described in terms of a gas of instanton quarks. It would be interesting to see what impact the corrections due to non-instanton solutions would have on the properties of this gas. Perhaps as a result of these corrections the properties of the gas in all $G(M, N)$ models would become more alike, thus providing some connection with the

results obtained in various $1/N$ expansions.

Are any of the results found in Grassmannian models also true in other models? Can one generalize the techniques discussed to studies of other models, say, CP^{N-1} in $2+1$ dimensions, or non-Abelian gauge theories? The preliminary answer to the first question seems positive; the answer to the latter one is that we do not know.

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